Competing Teams in Large Markets

Free Entry Equilibrium with (Sub-)Optimal Contracts*

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Abstract

In this paper, we formalize a market with a large number of competing production teams following Alchian and Demsetz (1974). We allow for wide-spread externalities which can affect players' payoffs. These externalities include changes in market conditions and pollutions, and may generate a variety of equilibrium outcomes. There are finite types of atomless players, who can form team-firms with finite memberships using available technologies. Given an arbitrary set of feasible partnership contracts for each team type, we consider free entry equilibrium as our equilibrium concept—in a free entry equilibrium, no team type can attract its members from other teams by proposing any implementable partnership contract. Furthermore, in a free entry equilibrium, players of the same type may have different payoffs—unequal treatment of equals. We show that as long as each firm type's implementable payoff set is compact and continuous in externality variables, there exists a free entry equilibrium. We provide several applications of our results.

Keywords: competing teams, widespread externalities, unequal treatment of equals, free entry equilibrium, labor managed firms, coalition formation

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1 Introduction

Market conditions affect firms' organizational forms such as their choice of technologies, incentive schemes, and ownership structures: a change in a product price can cause changes in firms' ownership structure and technology choice, and a change in agents' outside options provided by the markets certainly can affect firms' incentive schemes through participation constraints. Firms' organizational forms also affect market conditions: changes in ownership structures of firms cause changes in production efficiency, market prices, and agents' outside options. To analyze the simultaneous and endogenous nature of determining market conditions and firms' organizational forms in the markets, it is desirable to have a systematic tool that can describe the interactions between these two.

This paper considers a model in which small team producers (organizations) compete with one another in large markets. Each team is negligibly small, but their policy choice in the aggregate affects market conditions such as price and quantity, generating widespread externalities. Within a team its members choose their actions (effort levels) given the team's policy (contracts/partnership agreement), and given the market prices and the outside options (in payoffs) determined in the markets. In this sense, teams' organizations and actions influence one another only through changes in market conditions. We assume that there are finite types of atomless players whose types are observable. A team has a finite membership, each of its positions is assigned to a specific type of players, and those players engage in production by contributing their efforts to the team. We assume that the number of team types is also finite. With a continuum of players, there is a resulting continuum of teams that compete in the market. Thus, each team cannot have any influence on market conditions, while a change in overall team structure can.

For convenience, we consider two stages although we are not considering a multistage noncooperative game: in stage one, players are partitioned into teams with various policies to form a team structure, and in stage 2, players choose their actions optimally forming a market equilibrium allocation. Given a team structure, we consider deviations of players who propose to form a new team with a new policy as long as all members of it end in improved positions compared to the ones they currently belong to given the market conditions. We call a team structure a *free entry equilibrium* if there is no feasible team type with any feasible policy that can deviate from the team structure by attracting players to all positions of the team under the prevailing market conditions (for example, market prices; more generally, widespread externalities). That is, a free entry equilibrium endogenizes the organizations of teams and players' actions and payoffs, which determine market conditions, and market conditions affect the organization of teams. We will provide foundations for investigating what partnership/contract structure emerges in such an environment.

Free entry equilibrium is a natural adoption of a solution concept from atomless cooperative games (the f-core in Kaneko and Wooders 1986) applied to our industrial organization problem with endogenous production teams that offer incentive contracts to team members. Unlike most existing applications of the f-core, we consider competing (intrateam) suboptimal contracts in the presence of free-riding incentives and moral hazard problems.[1] We pinpoint the constrained optimal contracts with endogenous participation constraints as the surviving contracts when any feasible partnerships can enter and be utilized in the market. One unique finding of this approach is that due to coexisiting team contracts, in a free entry equilibrium the same type of players are not necessarily receiving the same payoffs: some players of the same type may be lucky to belong to teams that treat them well, but others may belong to teams that can only offer lower payoffs. In particular, if the available contracts are constrained with limited liability requirements or others, then the set of implementable payoff vectors may not satisfy comprehensiveness (freely disposable payoffs). Thus, even if a type of worker in a team is getting a higher payoff than their outside option, it may not be beneficial for the team to reduce her payoff. Also, although we maintain the assumption of finite team types, the structure of a type of teams can be complex enough to allow for subdivisions with various structures of ownership (decision making) rights. This allows us to investigate how vertical or lateral integrations emerge in markets following the literature of incomplete contracts (Grossman and Hart 1986, Hart and Moore 1990, and Hart and Holmstrom 2010), providing a general analytical tool for the literature of an Organizational Industrial Organization proposed by Legros and Newman (2013, 2014).

Our proof of the existence of free entry equilibrium is based on a standard fixed point theorem, utilizing convexification of atomless population measures. There are several difficulties to overcome. First, each team type's feasible policy set can be highly nonconvex due to the second best nature of the problem. Second, exactly due to the nonconvexity of feasible sets caused by limited liability constraints and others, it is not necessary for players of the same type to obtain the same payoffs even within the same teams. And third, even if the same type players are getting the same payoffs, they may affect the widespread externalities differently since externality variables are determined from the policies chosen by the teams. We will overcome the first issue by working on payoff space as Kaneko and Wooders (1986) did, and by using an indirect step.[2] We start by defining an "equal-treatment" free entry equilibrium which satisfies all other conditions for free entry equilibrium such as feasibility constraints, but in which players can dispose of payoffs if they are getting a higher payoff than other players of their type (comprehensive covers). Since such an allocation is still immune to entry by any team with any policy, our approach is to start by first finding an "equal-treatment" free entry equilibrium and then constructing the actual unequal treatment equilibrium by returning the disposed payoffs. The reason that we first impose an equal-treatment property is that a fixed-point approach is easily applicable by considering a product mapping assigning a weak-Pareto payoff vector to each team, while letting each type of

[1] With an exception of Legros and Newman (1996). See the literature review.

[2] Konishi and Simeonov (2025) provide a direct proof of the nonemptiness of the f-core without assuming comprehensiveness. For the first point, we use their approach.

players choose their favorite positions across team types.[3] For the third point, we need to connect each player type's payoff with team types and their policies. We work on the space of measures over the sets of policies, since the space of measures is well-behaved even if the sets of feasible policies are nonconvex and even disconnected. We can apply the Fan-Glicksburg fixed point theorem on the space of measures following the distribution approach by Mas-Colell (1984) and Jovanovich and Rosenthal (1988).[4]

The rest of the paper is organized as follows. In the next subsection, we provide a brief review of some relevant literature of the paper. In Section 2, we provide a simple Cournot market model with labor managed firms to illustrate our equilibrium concept, free entry equilibrium. In Section 3, we present the model, and introduce our equilibrium concepts formally. In Section 4, we prove a general existence theorem of free entry equilibrium by working on the sets of implementable allocations in payoff spaces. In Section 5, we conclude the paper with applications, including a model of households in markets, and large team contests with endogenous memberships.

1.1 A Brief Literature Review

Our free entry equilibrium is to require that no small individual team can successfully deviate from the equilibrium allocation. This approach was originally developed in Kaneko and Wooders (1986), which proved that the core of NTU characteristic function games with atomless players of finite types is nonempty when the cardinality of admissible coalitions is finite under very general conditions.[5] Hammond, Kaneko, and Wooders (1989) and Kaneko and Wooders (1989) extended this result by allowing for wide-spread externalities in the model in the context of an exchange economy with wide-spread externalities, and showed the equivalence between market equilibrium and the f-core, while the f-core and the standard Aumann core do not coincide. In contrast, our paper allows for suboptimal allocations for teams subject to moral hazard (free-riding) problems incentives, and the set of feasible payoff allocations for each team may not satisfy comprehensiveness. We prove the existence of free entry equilibrium by extending the proof of nonemptiness of f-core by Konishi and Simeonov (2025) by keeping track of widespread externalities caused by formed teams and their policies. Zame (2007) presented a comprehensive model of small teams in a general equilibrium model with many commodities and a price system under asymmetric information of not only moral hazard but also adverse

[3] This second "population mapping" ensures the weak Pareto optimality of the equilibrium allocation. Beato and Mas-Colell (1985) showed that even if firm productions are operated on the Pareto frontiers, the marginal cost pricing may not achieve (weak) Pareto optimality with nonconvex technologies. We avoid this problem with atomless players and the free mobility of players.

[4] Konishi and Simeonov (2025) prove the nonemptiness of the f-core without widespread externalities by using Kakutani's fixed point theorem. With widespread externalities we need to work on space of measures.

[5] In a TU setup, Wang (2020) analyzed f-core with a continuum of types of agents, and uncovered a link between f-core and transportation problem.

selection.[6] Unlike our model, Zame (2007) assumed a membership price system, which implies that players face the need to pay upfront membership payment regardless of what happens after they join their team: i.e., players are subject to unlimited liability. In contrast, we allow for limited liability of players, which can cause the same type players to be treated unequally if they join different types of teams.

Recently, two papers showed the existence of stable matchings in large two-sided matching markets using distribution approach. Geinecker and Kah (2021) proved the existence of a stable matching of the marriage problem with a continuum of types.[7] Carmona and Laohakunakorn (2024) considered many-to-one market, allowing for players' occupational choice (which side of the market they belong to). They construct their model skillfully so that they only need to check one side's (manager-side) deviation incentives to check stability of a matching. Allowing for multiple players belonging to a team, their paper is the most closely related to our model, but there are notable differences. First, their paper requires two-sided structure even though managers are the only players who can deviate. In contrast, our model can handle coalitions such as labor managed firms without having a single manager of a team without two-sided structures. Second, Carmona and Laohakunakorn (2024) allow for allocations that match one manager to finite workers such as school choice problem, or to a positive measure of workers.[8] In contrast, we focus on the former type finitely populated teams following the modeling by Kaneko and Wooders (1986): our definition of feasible assignments is based on their measure consistency assumption. Third, we restrict our attention to the case of finite types of players, although Carmona and Laohakunakorn (2024) allow for a very general player type space.

Konishi (2013) and Gersbach, Haller, and Konishi (2015) analyzed large markets when clubs (gated communities) and households (composed of multiple people) are endogenously formed and the members' choices are subject to local externalities. They proved existence of equilibrium by assuming that the members of clubs and households collectively choose a Pareto efficient allocation. In order to find a Pareto efficient allocation for the members of a club or

[6]Ellickson et al. (1999) proved the existence and optimality of large market equilibria with competing finite-membership efficient clubs, and Zame (2007) extended the former model by allowing for firms that are subject to moral hazard and adverse selection. Allouch, Conley and Wooders (2009) proved the existence and optimality of club equilibrium by using the results of nonemptiness of the f-core in the economy with wide-spread externalities by Hammond et al. (1989) and Kaneko and Wooders (1989).

[7]Greinecker and Kah (2021) consider the environment where every finite sample of the model admits a stable matching, while we do not. The approaches are significantly different, and it is not clear if we can use their approach to extend our model with an infinity of types of agents.

[8]Carmona and Laohakunakorn (2025) investigate how two similar knowledge-based theories by Rosen (1982) and Garicano and Rossi-Hansberg (2004) draw starkly different stable matchings using the model by Carmona and Laohakunakorn (2024). They conclude that the difference comes from the fact that Rosen (1982) allows each manager to be matched with unlimited number of workers, while Garicano and Rossi-Hansberg (2004) restrict the number of workers. Carmona and Laohakunakorn (2025) discuss this point using the flexibility of their model.

a household, they assume quasiconcavity of payoff functions for all externality variables and apply an existence theorem by Shafer and Sonnenschein (1975). In contrast, in this paper, we work on payoff space instead of allocation space, and we can drop quasiconcavity of payoff function completely. The strong advantage of this approach is that we do not need to select a (Pareto) optimal allocation for a club or a household. Although the second best set of allocations may be highly nonconvex, it could be much better-behaved in the payoff space. By inventing the way to keep track of actual allocations that achieve a payoff vector as an inverse mapping of payoff functions, we can prove the existence of free mobility equilibrium with continuous payoff functions and compact sets of feasible (implementable) allocations when there is no widespread externality. With widespread externalities, we need somewhat restrictive additional constraints: in each team type, the set of implementable allocations in payoff space is a continuous correspondence.[9]

This paper is also closely related to both organization economics and industrial organization. In their influential paper, Alchian and Demsetz (1972) introduced the notion of team production in which several complementary resources are used as input, and argued how a team organization problem can be created depending on the ownership of the resources. Holmstrom (1982) proved that the first best allocation cannot be achieved in general if all profits need to be distributed within a team (partnership), while it can be done if there is a residual claimant (principal-agent relationship). Starting from Grossman and Hart (1986) and Hart and Moore (1990), the literature of incomplete contracts analyzed how firm boundaries are determined by defining asset ownership and control rights. Departing from single firm's problems, Legros and Newman (1996) considered many small team producers in a market, and showed that different types (in their endowments) of agents participate in different forms of contracts in equilibrium. However, they did not consider how team organizations affect market conditions. Recently, Legros and Newman (2013, 2014) advocate an organizational industrial organization (OIO), emphasizing that firms' internal organizations can be important factors that determine firms' conduct and market conditions such as market price, quantity, and welfare.[10] Our current paper considers formation of organizational teams with widespread externalities

[9]What it means is that it is not sufficient to assume that the set of implementable allocations is an upper hemicontinuous—each type of feasible contracts should not vanish suddenly with a change in widespread externalities. Zame (2007) avoided this problem by assuming the set of feasible contracts is ex ante given, and is finite. Legros and Newman (1996) considered optimal contract as a function of market and cost parameters, but they do not endogeneize these parameters as wide-spread externalities. We will need continuity of implementable policies, which could be a problem in generalizing the approach by Legros and Newman (1996) in our setting.

[10]Hart and Holmstrom (2010) considered a model in which final products are produced with two complementary inputs provided by two suppliers with noncontractible production decisions, and analyzed the organizational problem arisen from a conflict of interests of input suppliers. Using a perfectly competitive market with the Hart-Holmstrom-type organizational problem, Legros and Newman (2013) showed that organizational forms and market price interact with each other, and heterogeneous organizations can coexist differing in their performance.

including market conditions, thus it could provide a useful theoretical foundation for the OIO models.

Finally, we relate this paper to a companion paper on stable team structure written by the same authors. Konishi, Pan, and Simeonov (2025) consider a Tullock contest played in an L team sport league, where each team has players in M positions and they engage in contributing efforts to win a prize. The winning team distributes the prize among the team member players according to a sharing rule based on their positions. In this problem, stable team structures of the league are analyzed when players can switch their teams (and positions) through headhunting.[11] Konishi et al. (2025) compare egalitarian and highly differential sharing rules, and demonstrate that there are tradeoffs between intra and inter-team inequalities. Unfortunately, it is not easy to show a general existence of stable team structure due to the discreteness of the problem. Here, our approach provides a remedy by replicating the leagues.

2 An Example: Labor Managed Firms

In order to illustrate our model and equilibrium concept (free entry equilibrium), we present a Cournot market example with labor managed firms (a simple partnership contract). Consider two types of workers 1 and 2 with population mass $\bar{\nu}_1$ and $\bar{\nu}_2$ with $\bar{\nu}_1 = 1 < \bar{\nu}_2 = 2$, and a CES team production technology $f(e_1, e_2) = \left(e_1^{\frac{1}{3}} + e_2^{\frac{1}{3}}\right)^2$, where $e_i \geq 0$ is worker i's effort in a firm $i = 1, 2$. This means that there are complementarities between these two workers' efforts. Workers' cost of making effort is linear: $c_i(e_i) = e_i$. Each team firm can choose its proportional sharing rule (θ_1, θ_2). The inverse demand function is described by $p = 1 - Y$. Under the market price p, if a firm's sharing rule is (θ_1, θ_2), worker $i = 1, 2$ solves:

$$\max_{e_i} \theta_i p f(e_i, e_j) - e_i,$$

and worker i's equilibrium payoff and output level are (see Appendix):

$$u_i = \left\{ \frac{4\theta_i}{9} \left(\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}}\right)^4 - \frac{8\theta_i^{\frac{3}{2}}}{27} \left(\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}}\right)^3 \right\} \times p^3,$$

and

$$y = \left\{ \frac{4}{9} \left(\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}}\right)^4 \right\} p^2,$$

respectively. Note that both u_i and y are multiplicatively separable in p. Plotting (u_1, u_2) in the payoff profile space for all $\theta_1 \in [0, 1]$, we have the red curve (adjusted by $\frac{1}{p^3}$) (see Figure 1). When $\theta_1 = 1$ ($\theta_2 = 0$), the equilibrium payoff allocation is A_1, and it moves along the curve as θ_1 decreases (θ_2 increases),

[11] Kobayashi, Konishi, and Ueda (2025) consider a single team's optimal sharing rule in a generalized group contest problems with effort complementarities without considering players' mobility.

and reaches A_2 when $\theta_1 = 0$ ($\theta_2 = 1$). Note that there are trivial variants of the above technology: solo teams without a partner. Figure 1 includes these cases as they correspond to A_1 (type 1 only) and A_2 (type 2 only).

We first analyze a feasible allocation that satisfies an equal treatment property, which we may call a *free mobility equilibrium*. Since $\bar{\nu}_1 = 1 < \bar{\nu}_2 = 2$, measure $\bar{\nu}_2 - \bar{\nu}_1 = 1$ type 2 players get the solo team payoff $\underline{u}_2 = 0.148p^3$ at point A_2. Thus, to be a free mobility equilibrium, any type 2 player in a two person team should also get $\underline{u}_2$. In order to achieve this, two person teams achieve point B by choosing $(\theta_1^{fm}, \theta_2^{fm}) = (0.857, 0.143)$. Thus, a measure 1 of type 2 workers choose A_2, and a measure 1 of pairs of types 1 and 2 choose B in the free mobility equilibrium.

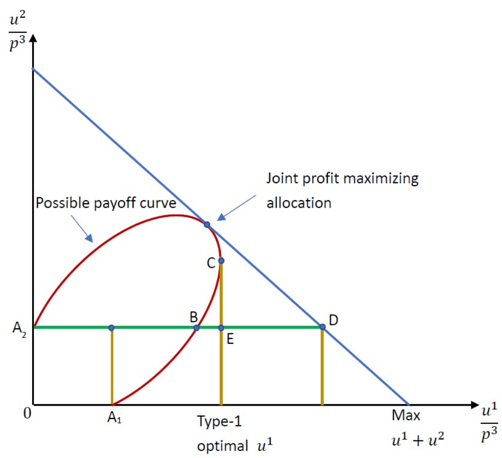

Figure 1

Now, we turn to our equilibrium concept: a *free entry equilibrium* requires that there is no firm with a feasible policy (sharing rule) that can break the equilibrium partenrship strucuture by attracting players to all positions of the team. Is the above free mobility equilibrium a free entry equilibrium? It is not. Workers in a team firm can be better off by reducing θ_1 to incentivize type 2 workers, since at B the curve is still upward sloping. Let us calculate θ_1 that maximizes u_1. Solving $\frac{\partial u_1}{\partial \theta_1} = 0$, we obtain $(\theta_1^{1opt}, \theta_2^{1opt}) = (0.687, 0.313)$, and the resulting allocation is point C: $(u_1^{1opt}, u_2^{1opt}) = (0.683p^3, 0.378p^3)$. Since $(u_1^{1opt}, u_2^{1opt}) > (u_1^{fm}, u_2^{fm})$, both players in the team are better off in a Pareto manner by reducing θ_1 from $\theta_1^{fm} = 0.857$ to $\theta_1^{1opt} = 0.687$ (see Table 1).

(θ_1,θ_2)	y	(u_1,u_2)	
$(1,0)$	$0.44p^2$	$(0.148p^3,0)$	A_1 (solo)
$(0.857,0.143)$	$1.28p^2$	$(0.580p^3,0.148p^3)$	B
$(0.687,0.313)$	$1.65p^2$	$(0.683p^3,0.378p^3)$	C
$(0.5,0.5)$	$1.78p^2$	$(0.593p^3,0.593p^3)$	surplus max

Table 1: equilibrium payoff curve

Although there are still measure 1 of type 2 players who receive payoff $0.148p^3$ by working alone, the other measure 1 of type 2 players receive $0.378p^3$ by deviating from the free mobility equilibrium. This is the free entry equilibrium payoff profile (points A_2 and C). Despite this apparent violation of equal-treatment property, no firm can enter profitably (no further deviations), since all type 1 players are already getting the highest possible payoffs, and they will not be attracted by any other offers. Can we have any other free entry equilibrium? The answer is no, since if two-person teams are not choosing the type 1-optimal sharing rule, there will be teams entering the market with the type 1-optimal sharing rule, attracting type 1 players employed by two-person teams and self-employed type 2 players. Thus, in this example, free entry equilibrium is weakly Pareto efficient, unique, and does not satisfy equal treatment of equals.

So far, we took market price p as given, focusing on individual firms' behaviors.[12] However, a firm's sharing rule (policy) (θ_1,θ_2) affects the firm's output level y. Thus, as many firms adopt a new policy, the market equilibrium price will be affected through a change in the total output level Y through inverse demand $p = 1 - Y$. For example, moving from free mobility equilibrium allocation (points A_2 and B) to free entry equilibrium allocation (points A_2 and C) increases the total output and reduces the market price. The following table compares equilibrium payoff vectors taking this effect into account.

	Y	p	$(u_1^{team},u_2^{team},u_2^{solo})$
solo only	$0.88p^2$	0.569	0.0272
free mobility	$1.72p^2$	0.525	$(0.0839,0.0214,0.0214)$
free entry	$2.09p^2$	0.493	$(0.0818,0.0453,0.0177)$
transferrable utility	$2.22p^2$	0.483	$(0.117,0.0167,0.0167)$

Table 2: with wide-spread externalities

An interesting observation is the force of wide-spread externalities regarding the market price of the product. In a free mobility equilibrium, team production is inefficient due to the uneven sharing rule for type 1. So, each team has an incentive to revise their sharing rule to improve their payoffs in a Pareto manner. However, as all teams revise their sharing rules, the market output level increases, dropping the market price. As a result, type 1 workers' payoffs eventually goes down due to this effect.[13]

[12] Note that in this example, players' payoffs are affected by market price that is conveniently multiplicatively separable. This property does not hold in general, but we can find a free entry equilibrium in a different way.

[13] It is similar to the Braess paradox in transportation economics.

Lastly, we highlight a difference between our equilibrium concept and the ones in the general equilibrium literature. What if team members can transfer their payoff (transferrable utility) in the form of type-dependent (positive and/or negative) membership fees? That is, before a production team is formed, the team requires from its workers a membership fee (and a sign-up fee or bonus) to join the team, and then, workers play their effort contribution game. Since our economy is quasi-linear, the membership fees do not affect the workers' equilibrium effort contributions for the same (θ_1, θ_2). Since the pie is maximized at $\theta_1 = \theta_2 = \frac{1}{2}$, the total payoff is $0.593p^3 \times 2 = 1.186p^3$. Since there are type 2 workers getting $\underline{u}_2 = 0.148p^3$, the equilibrium membership fee for type 2 workers is $T_2 = 0.593p^3 - 0.148p^3 = 0.445p^3$, which will be used as a sign up bonus (membership subsidy) for type 1 workers: $-T_1 = 0.445p^3$. This is point D, which is the equilibrium by Zame (2007). Thus, type 2 workers get $\underline{u}_2 = 0.148p^3$, irrespective of the types of firms they belong to, while type 1 workers get $u_1 = 0.593p^3 + 0.445p^3 = 1.038p^3$ (points A_2 and D).

3 The Model

Consider a general team formation model with widespread externalities. Widespread externalities p can be market prices, pollution, or other forms of externalities such as distributions of teams to be matched with. We assume that $\mathbb{P}$ is a compact subset of an L-dimensional Euclidean space $\mathbb{P} \subset \mathbb{R}^L$, and $p \in \mathbb{P}$ is taken as given by each player and team. Teams can engage in activities by recruiting players who can provide heterogeneous services (actions and labor efforts) depending on their types (distinguished by their ability, marginal cost of effort etc). Within each team players engage in a game (an effort contribution game or some other game). We assume that the set of player types T is finite and that each type $t \in T$ has a continuum of players with a Borel measure $\bar{\nu}^t > 0$. The total population of players has measure $\sum_{t \in T} \bar{\nu}^t$. We assume that players' types are *observable* by teams (or other team members). Thus, there is no adverse selection problem in our model.

First, we define a team type γ as a list of technology $f \in \mathcal{F}$, finite positions M^f, each to be filled by a single player, and a task assignment function $\alpha : \{1, ..., M^f\} \to T$ specifying which type of player is to be assigned to each position. We assume that $\mathcal{F}$ is a finite set. Due to technological constraints, each position is open to a subset of types of players (for example, technical positions are only for technical player types). This limits the set of feasible assignments α for technology $f \in \mathcal{F}$.[14] A representative team type γ is a pair (f, α), and the set of all feasible team types is denoted by Γ.[15] For convenience, team type γ's technology and assignment rule are denoted by $\gamma \equiv (f^\gamma, \alpha^\gamma)$ with $M^\gamma = M^{f^\gamma}$. Since both T and $\mathcal{F}$ are finite, this ensures that the set of team types Γ is also

[14] If position m of technology f can be assigned to player types t and t', then we prepare two different assignment function α and α' with $\alpha(m) = t$ and $\alpha'(m) = t'$.

[15] Even with the same technology f, if two different assignment rules α and α' are used (i.e., $\alpha(m) \neq \alpha'(m)$ for some $m = 1, ..., M^f$), then (f, α) and (f, α') are two different team types.

finite.

Each team of team-type $\gamma \in \Gamma$ chooses a policy $\theta^\gamma \in \Theta^\gamma$, which is a compact subset of an Euclidean space. A distribution of teams of type γ over their allocations is described by a Borel measure μ^γ over Θ^γ. Thus, the total measure of teams of type γ is $\mu^\gamma(\Theta^\gamma)$, and the total measure of teams is $\sum_{\gamma\in\Gamma}\mu^\gamma(\Theta^\gamma)$. We define a team distribution to be the full profile of these measures $\mu = (\mu^\gamma)_{\gamma\in\Gamma}$. Let $\mathbb{M}^\gamma$ be the set of Borel measures on Θ^γ, and $\mathbb{M} \equiv \prod^\gamma_{\in\Gamma}\mathbb{M}$ be the set of all measure profiles on $\Theta \equiv \prod_{\in\Gamma}\Theta^\gamma$.

Let $\mathcal{M}^\gamma = \{(m,\gamma) : m \in \{1,...,M^\gamma\}\}$ be the set of all positions of team type $\gamma \in \Gamma$, and let $\mathcal{M} = \bigcup_{\gamma\in\Gamma}\mathcal{M}^\gamma$ be the set of all positions across all admissible team types. Also, let $\mathcal{M}^t = \{(m,\gamma) : \alpha^\gamma(m) = t \text{ for some } \gamma \in \Gamma, (m,\gamma) \in \mathcal{M}^\gamma\}$ be the subset of all positions occupied by type t players. Clearly, $\{\mathcal{M}^t\}_{t\in T}$ is a partition of $\mathcal{M}$. For each $(m,\gamma) \in \mathcal{M}$, let $t(m,\gamma)$ be the player type that is assigned to position m: i.e., $\alpha^\gamma(m) = t$. *We assume that for each type $t \in T$, there is a trivial single-member team type $\gamma^t \in \Gamma$ such that $M^{\gamma^t} = 1$ and $\alpha^{\gamma^t}(1) = t$. The payoff obtained from joining γ^t can be regarded as an outside option for type t.* For convenience, let $\Gamma_{sngl} \equiv \{\gamma^t\}_{t\in T}$ be the set of all singleton teams, and let $\Gamma_{team} \equiv \{\gamma \in \Gamma : M^\gamma \geq 2\}$ be the set of multiple-member teams.[16] Obviously, $\{\Gamma_{sngl}, \Gamma_{team}\}$ is a partition of Γ.

For each $\gamma \in \Gamma_{team}$, and each $m \in \{1,...,M^\gamma\}$, the distribution of type $t(m,\gamma)$ players over team type γ's policies Θ^γ is described by a measure ν^γ_m over Θ^γ: i.e., $\nu^\gamma_m(\Theta^\gamma)$ is the measure of type $t(m,\gamma)$ players who occupy team-type γ's mth position.

Finally, we adopt "measure consistency" assumption in Kaneko and Wooders (1986, pp. 108-109), which requires that population measure of players is preserved in one-to-one functions in order to make sense economically. We will build this requirement in the definition of feasible assignments.

Definition 1. A **feasible assignment** is a list $\left((\nu^\gamma_m)_{m\in\mathcal{M}^\gamma,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$ such that (i) $\sum_{(m,\gamma)\in\mathcal{M}^t}\nu^\gamma_m(\Theta^\gamma) = \bar{\nu}^t$ for all $t \in T$, and (ii) for any measurable subset $S \subseteq \Theta^\gamma$, $\nu_1(S) = ... = \nu_{M^\gamma}(S) = \mu^\gamma(S)$ holds for all $\gamma \in \Gamma$.

The interpretation of this assignment is that there is a measure μ^γ of type γ teams hiring a measure $\nu^\gamma_m = \mu^\gamma$ of players for each position $m \in \mathcal{M}^\gamma$. The feasibility of the assignment is described by (i), the market balance condition for each type of players.

In order to describe feasible allocations, we introduce a few more definitions. We do not specify a concrete game played in team type γ under policy θ^γ. Instead, we assume that (1) for each team type $\gamma \in \Gamma$, and each $m = 1,...,M^\gamma$, there is a continuous payoff function $u^\gamma_m : \Theta^\gamma \times \mathbb{P} \to \mathbb{R}$ for type $t = t(m,\gamma)$ players; (2) there is a feasible policy correspondence $Z^\gamma : \mathbb{P} \twoheadrightarrow \Theta^\gamma$, which is nonempty-valued, compact and continuous; and (3) there is a widespread externality function $\varphi : \mathbb{M} \times \mathbb{P} \to \mathbb{P}$ that is continuous.[17] Let $u^\gamma : \Theta^\gamma \times \mathbb{P} \to \mathbb{R}^{\mathcal{M}^\gamma}$

[16] Teams with homogeneous type of players belong to Γ_{team}.

[17] Extending this to a correspondence, we can cover market price vector as a widespread (pecuniary) externalities: see the application section 5.2.

be team γ's payoff profile function such that $u^\gamma(\theta^\gamma, p) \equiv (u_{t(m,\gamma)}(\theta^\gamma, p))_{m=1}^{M^\gamma}$. Clearly u^γ is a continuous function.

Definition 2. A **feasible allocation** is a list $\left(p, (\nu_m^\gamma)_{m \in \mathcal{M}^\gamma, \gamma \in \Gamma}, (\mu^\gamma)_{\gamma \in \Gamma}\right)$ such that (i) $\left((\nu_m^\gamma)_{m \in \mathcal{M}^\gamma, \gamma \in \Gamma}, (\mu^\gamma)_{\gamma \in \Gamma}\right)$ is a feasible assignment, (ii) $\mu^\gamma(\Theta^\gamma \backslash Z^\gamma(p)) = 0$ for all $\gamma \in \Gamma$, and (iii) $p = \varphi(\mu, p)$ holds.

Our main equilibrium concept in this paper is a free entry equilibrium in which no new team can attract all workers needed to fill the positions with any feasible contract/policy. The next equilibrium concept takes the following equilibrium output and equilibrium payoff allocation mappings $u^\gamma : \Theta^\gamma \times \mathbb{P} \to \mathbb{R}_+^{M^\gamma}$ as given.

Definition 3. A feasible allocation $\left(p, (\nu_m^\gamma)_{m \in \mathcal{M}^\gamma, \gamma \in \Gamma}, (\mu^\gamma)_{\gamma \in \Gamma}\right)$ is a **free entry equilibrium** if there is no pair $(\gamma, \theta^\gamma) \in \Gamma \times Z^\gamma(p)$ such that for all $m \in \mathcal{M}^\gamma$, there are $(m', \gamma') \in \mathcal{M}^{t(m,\gamma)}$, and $S \subset Z^{\gamma'}(p)$ with $\mu^{\gamma'}(S) > 0$ such that $u_m^\gamma(\theta^\gamma, p) > u_{m'}^{\gamma'}(\theta^{\gamma'}, p)$ in S almost everywhere.

The reason that we require $\mu^{\gamma'}(S) > 0$ in the above definition is to make sure that an equilibrium allocation is immune to coalitional deviations that can improve a non-negligible number of players. Note that the definition of free entry equilibrium does not require that the same type players receive identical payoffs in equilibrium.[18]

4 Existence of Free Entry Equilibrium

Here, we will prove the existence of free entry equilibrium. As we discussed in the introduction, it is more convenient to work on payoff spaces of player types than on teams' policy spaces. This is because the set of each team type's feasible policy set is highly nonconvex, while (a comprehensive cover of) the weak Pareto frontier in the payoff space is essentially homeomorphic to a simplex that is compact and convex.[19] Obviously, it is not enough for us to work on the sets of weak Pareto frontiers, since we would like to allow for limited liability constraints, which may not permit bringing down payoffs by simply disposing them (see Figure 1). However, there is a remedy for this. We consider hypothetical allocations that allow for free disposal of payoffs keeping all other feasibility requirements intact. Then we find an equal treatment free entry equilibrium in this expanded feasible set. What we need is that there is no potential team type that can improve on the desired allocation strictly. Thus, if we can assign

[18]As is seen from our example, there may not be feasible equal treatment allocation.

[19]This method has been used in Konishi and Simeonov (2025) in proving nonemptiness of the f-core in an atomless characteristic function form game introduced by Kaneko and Wooders (1986). In contrast, here we work on a model that is readily applicable to a variety of concrete economic problems with widespread externalities, which requires us to keep track of the policy distributions of all team types.

a weakly Pareto efficient allocation to each potential team type γ with this expanded feasibility in which players are choosing their most favorite positions, then such an allocation is immune to any potential entry of a team type with (strictly-improving) feasible policy. Thus, if we find a feasible allocation that is a fixed point of such position choice problem, then it would be a (locally) weakly Pareto-undominated free entry equilibrium.[20] To see this, note that for a team-type γ to improve upon such an allocation, it must prepare a strictly higher payoff than the equilibrium payoff for each of its members. Since every team type γ offers a weakly Pareto efficient payoff vector, there cannot be strictly improving deviation from such an allocation, and it must be a free entry equilibrium.

In the following, we expand the notion of feasibility by allowing for disposal of payoffs (comprehensiveness) as a device. With this device, the weak Pareto frontier of team type γ becomes connected and is contractible in its interior so that we can apply a fixed point theorem easily.

Definition 4. An **equal-treatment allocation under comprehensiveness** $\left(p, (u_t^*)_{t\in T}, (\nu_m^\gamma)_{m\in\mathcal{M}^\gamma,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$ is a pair of a payoff profile $(u_t^*)_{t\in T}$ and a feasible allocation $\left(p, (\nu_m^\gamma)_{m\in\mathcal{M}^\gamma,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$ such that for all types $t \in T$, $\sum_{(m,\gamma)\in\mathcal{M}^t} \nu_m^\gamma(\{\theta^\gamma \in Z^\gamma(p) : u_t(\theta^\gamma, p) \geq u_t^*\}) = \bar{\nu}^t$ holds. Such a payoff profile $(u_t^*)_{t\in T}$ is called an **equal treatment payoff profile**.

Note that if $(u_t^*)_{t\in T}$ is an equal treatment payoff profile, any $(u_t')_{t\in T} \leq (u_t^*)_{t\in T}$ is an equal treatment profile. This proves convenient for Lemma 2 below. Obviously, only the highest equal treatment payoff profile will eventually matter in constructing a free entry equilibrium.

Definition 5. An **equal-treatment free entry equilibrium under comprehensiveness** is an equal treatment allocation under comprehensiveness $\left(p, (u_t^*)_{t\in T}, (\nu_m^\gamma)_{m\in\mathcal{M}^\gamma,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$ such that there is no pair (θ^γ, γ) with $\gamma \in \Gamma$ and $\theta^\gamma \in \Theta^\gamma$, for which $u_m^\gamma(\theta^\gamma, \gamma) > u_{t(m,\gamma)}^*$ holds for all $m = 1, ..., M^\gamma$.

Once the existence of an equal-treatment free entry equilibrium under comprehensiveness is proven, it also implies that there is a free entry equilibrium (see Definition 3). In Figure 1, an equal-treatment free entry equilibrium under comprehensiveness corresponds to points A_2 and E, and a free entry equilibrium corresponds to points A_2 and C. If there is an equal treatment free entry equilibrium under comprehensiveness, then clearly (i) there is a (unequal-treatment) feasible allocation, and (ii) it is immune to improvement by any team type γ via any θ^γ, since in such a feasible allocation, all types of players are getting at least $(u_t^*)_{t\in T}$ almost everywhere. Thus, it is automatically a free entry equilibrium.

We first define feasible payoff allocations for each team type $\gamma \in \Gamma_{team}$ by keeping track of feasible policies that achieve each feasible payoff vector.[21] Let

[20] With widespread externalities, even if there is no profitable deviations by finite teams given p, there may be a profitable positive measure deviation coordinated by a mass of teams.

[21] For a singleton teams $\gamma^t \in \Gamma_{sngl}$, the maximum payoff level for the member $u^{\gamma^t}(p)$ is uniquely determined.

team type γ's weak Pareto policy correspondence $\Theta^\gamma_{WP} : \mathbb{P} \twoheadrightarrow \Theta^\gamma$ be such that $\Theta^\gamma_{WP}(p) \equiv \{\theta^\gamma \in Z^\gamma(p) : \nexists \theta^{\gamma\prime} \in Z^\gamma(p)$ such that $u^\gamma(\theta^{\gamma\prime}, p) \gg u^\gamma(\theta^\gamma, p)\}$. Similarly, let $V^\gamma_{WP} : \mathbb{P} \twoheadrightarrow \mathbb{R}^{M^\gamma}$ be team type γ's weak Pareto payoff correspondence such that $V^\gamma_{WP}(p) \equiv u^\gamma(\Theta^\gamma_{WP}(p))$. Let $\bar{V}^\gamma : \mathbb{P} \twoheadrightarrow \mathbb{R}^{M^\gamma}$ be a comprehensive cover correspondence of V^γ_{WP} such that $\bar{V}^\gamma(p) \equiv \left\{u^\gamma \in \mathbb{R}^{M^\gamma}_+ : u^\gamma \leq \tilde{u}^\gamma \text{ for some } \tilde{u}^\gamma \in V^\gamma_{WP}(p)\right\}$ for all $p \in \mathbb{P}$. (See Figure 2.)

We will need to treat singleton teams separately for a normalization purpose. A singleton team $\gamma^t \in \Gamma_{sngl}$ has a unique maximum payoff $u^{\gamma^t *}(p) \equiv \max_{\theta^{\gamma^t} \in Z^{\gamma^t}(p)} u^{\gamma^t}(\theta^{\gamma^t})$, $\Theta^\gamma_{WP}(p) \equiv \arg\max_{\theta^{\gamma^t} \in Z^{\gamma^t}(p)} u^{\gamma^t}(\theta^{\gamma^t})$, and $V^{\gamma^t}_{WP}(p) \equiv u^{\gamma^t *}(p)$. Since Z^{γ^t} is a continuous correspondence and u^{γ^t} is a continuous function, $V^{\gamma^t}_{WP}(p) \equiv u^{\gamma^t *}(p)$ is a continuous function, $\Theta^\gamma_{WP}(p)$ is a nonempty-valued and upper hemicontinuous.

From now on, we will denote a K-dimensional simplex by Δ^K.

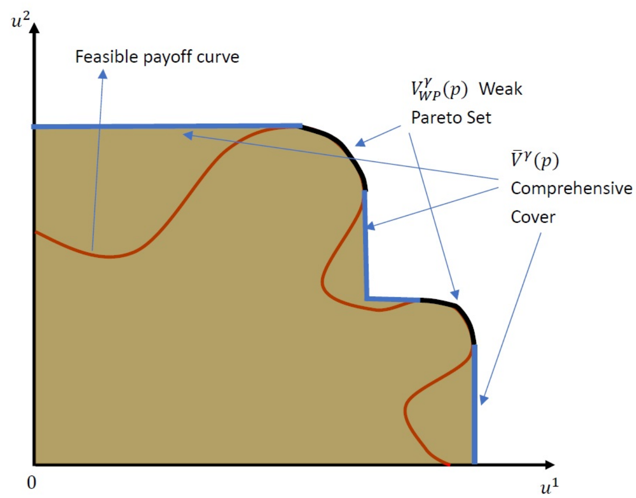

Figure 2

Lemma 1. Suppose that for all $\gamma \in \Gamma_{team}$, and all positions m of team type γ, type $t = t(m, \gamma)$'s indirect payoff function $u^\gamma_m(\theta^\gamma, p)$ is continuous in (θ^γ, p). Then, $\Theta^\gamma_{WP} : \mathbb{P} \twoheadrightarrow \Theta^\gamma$ has a closed graph, and $\bar{V}^\gamma : \mathbb{P} \twoheadrightarrow \mathbb{R}^{M^\gamma}$ is a continuous correspondence.

Proof. We first show that $\Theta^\gamma_{WP}(p)$ has a closed graph. Pick any $\left\{p^k\right\}^\infty_{k=1} \to \bar{p}$ and any $\left\{\theta^{\gamma k}\right\}^\infty_{k=1}$ such that $\theta^{\gamma k} \in \Theta^\gamma(p^k)$ for each $k = 1, 2, \ldots$ Since Θ^γ is a compact set, we can select a convergent subsequence of $\left\{\theta^{\gamma k}\right\}^\infty_{k=1}$. Relabeling k's, we have $\left\{p^k\right\}^\infty_{k=1} \to \bar{p}$ and $\left\{\theta^{\gamma k}\right\}^\infty_{k=1} \to \bar{\theta}^\gamma$, where $\theta^{\gamma k} \in \Theta^\gamma_{WP}(p^k)$ for each $k = 1, 2, \ldots$. Then, for each k, for any $\theta^{\gamma k\prime} \in \Theta^\gamma_{WP}(p^k)$, there is $m \in M^\gamma$ such that $u^\gamma_m(\theta^{\gamma k\prime}, p^k) \leq u^\gamma_m(\theta^{\gamma k}, p^k)$. Since M^γ is finite, we have by continuity

$\lim_{k\to\infty} u^\gamma_m(\theta^{\gamma k\prime}, p^k) = u^\gamma_m(\bar\theta^{\gamma\prime}, \bar p) \le \lim_{k\to\infty} u^\gamma_m(\theta^{\gamma k}, p^k) = u^\gamma_m(\bar\theta^\gamma, \bar p)$ for some $m \in M^\gamma$. This proves that the weak Pareto policy correspondence $\Theta^\gamma_{WP} : \mathbb{P} \twoheadrightarrow \Theta^\gamma$ has a closed graph property.

By continuity of u^γ and the closed graph property of Θ^γ_{WP}, V^γ_{WP} has a closed graph property as well.

Since V^γ_{WP} has a closed graph, V^γ_{WP} is upper hemicontinuous and $\bar V^\gamma$ is upper hemicontinuous. We can show that $\bar V^\gamma$ is lower hemicontinuous as well. Suppose that $u^\gamma \in \bar V^\gamma(\bar p)$, then there is $\tilde u^\gamma \in V^\gamma_{WP}(\bar p)$ such that $\tilde u^\gamma \ge u^\gamma$. Let $e^\gamma \equiv \tilde u^\gamma - u^\gamma \ge 0$. Let $\{p^k\}_{k=1}^\infty \to \bar p$. Since $\tilde u^\gamma \in V^\gamma_{WP}(\bar p)$, there are $\bar\theta^\gamma \in \Theta^\gamma_{WP}(\bar p)$ and $\{\theta^{\gamma k}\}_{k=1}^\infty \to \bar\theta$, then by continuity of u^γ, there is a sequence $\{\tilde u^{\gamma k}\}_{k=1}^\infty \to \tilde u^\gamma$ by setting $\tilde u^{\gamma k} = u^\gamma(\theta^{\gamma k}, p^k)$. Since $\bar V^\gamma(p) = V^\gamma_{WP}(p) + \mathbb{R}^{M^\gamma}_-$, by setting $u^{\gamma k} = \tilde u^{\gamma k} - e^\gamma \in \bar V^\gamma(p^k)$ for all $k = 1, 2, ...$, we have $\{u^{\gamma k}\}_{k=1}^\infty \to u^\gamma$. Hence, $\bar V^\gamma$ is lower hemicontinuous, thus $\bar V^\gamma$ is a continuous correspondence.■

Let $\bar\theta^\gamma_{WP} : \mathbb{R}^{M^\gamma} \times \mathbb{P} \twoheadrightarrow \Theta^\gamma$ with $\bar\theta^\gamma_{WP}(u^\gamma, p) \equiv \{\theta^\gamma \in \Theta^\gamma_{WP}(p) : u^\gamma(\theta^\gamma, p) \ge u^\gamma\}$ be a weakly Pareto efficient policy mapping that achieves payoff profile u^γ or higher. This mapping connects the payoff space and the policy space, showing which policies can achieve each feasible payoff allocation. It is important since market price p is affected by the policies adopted by teams.

Lemma 2. Suppose that for all $\gamma \in \Gamma_{team}$, and all positions m of team type γ, type $t = t(m, \gamma)$'s indirect payoff function $u^\gamma_m(\theta^\gamma, p)$ is continuous in (θ^γ, p). Then, $\bar\theta^\gamma_{WP} : \mathbb{R}^{M^\gamma} \times \mathbb{P} \twoheadrightarrow \Theta^\gamma$ is nonempty-valued and has a closed graph.

Proof. Note that $\bar\theta^\gamma_{WP}(u^\gamma, p) \equiv \{\theta^\gamma \in \Theta^\gamma : u^\gamma(\theta^\gamma, p) \ge u^\gamma\} \cap \Theta^\gamma_{WP}(p)$. Since u^γ is a continuous function and Θ^γ_{WP} has a closed graph, $\bar\theta_{WP}$ has a closed graph.■

We can regard $\left(\left(\bar V^\gamma(p)\right)_{\gamma\in\Gamma}, (\bar\nu_t)_{t\in T}\right)$ as an atomless nontransferable utility game given $p \in \mathbb{P}$. Let $\partial \bar V^\gamma(p) \equiv \{u^\gamma \in \bar V^\gamma(p) : \nexists \tilde u^\gamma \in \bar V^\gamma(p) \ s.t.\ \tilde u^\gamma \gg u^\gamma\}$ be the weak Pareto frontier of $\bar V^\gamma(p)$.

We will normalize $\left(\left(\bar V^\gamma(p)\right)_{\gamma\in\Gamma}, (\bar\nu_t)_{t\in T}\right)$ utilizing singleton teams' payoffs. For each singleton team $\gamma^t \in \Gamma_{sngl}$, set $\hat u^{\gamma^t}(p) = 1$ for all $t \in T$ and all $p \in \mathbb{P}$. For all $\gamma \in \Gamma_{team}$, all $m = 1, ..., M^\gamma$, and all $\theta^\gamma \in Z^\gamma(p)$, define $\hat u^\gamma_m(\theta^\gamma, p) \equiv u^\gamma_m(\theta^\gamma, p) - u^{\gamma^{t(m,\gamma)}}(p) + 1$, and rename it $u^\gamma_m(\theta^\gamma, p)$ for normalization so that $u^\gamma_m(\theta^\gamma, p) \ge 1$ is the individual rational payoffs for type $t = t(m, \gamma)$. This is a continuous function by Lemma 2. We set each player's individual rational payoff at 1 to ensure the existence of allocations that attain less payoffs than the individually rational level in the positive orthant of each team γ's payoff space. For all $\gamma \in \Gamma_{team}$, let $\bar V^\gamma(p) = \mathbb{R}^{M^\gamma}_-$ if $\bar V^\gamma(p) \cap \mathbb{R}^{M^\gamma}_+ = \varnothing$ holds. Note that no type t wants to choose such $\gamma \in \Gamma_{team}$, since a singleton γ^t dominates γ, noting $u^{\gamma^t}(p) = 1$.

Although we will work on a weak Pareto frontier of team type $\gamma \in \Gamma_{team}$ to assign a weakly Pareto efficient payoff vector to γ, the weak Pareto frontier itself can be highly nonconvex, and it may not be easy to work with. Thus, we introduce an **abstract policy set** $X^\gamma \equiv \Delta^{M^\gamma}$, which is compact and convex, and its interior set $intX^\gamma \equiv \{x^\gamma \in X^\gamma : x^\gamma \gg 0\}$. We have the following result.

Lemma 3. For all $\gamma \in \Gamma_{team}$, there exists a continuous function $w^\gamma : intX^\gamma \times \mathbb{P} \to \mathbb{R}^{M^\gamma}_{++}$ such that for each $p \in \mathbb{P}$ with $\bar{V}^\gamma(p) \neq \{0\}$, $w^\gamma(\cdot, p) : intX^\gamma \to \partial\bar{V}^\gamma(p) \cap \mathbb{R}^{M^\gamma}_{++}$ is one-to-one and onto function.

Proof. Let an artificial welfare function $h^\gamma : \mathbb{R}^{M^\gamma}_{+} \times intX^\gamma \to \mathbb{R}_+$ with welfare weights $\left(\frac{1}{x^\gamma_m}\right)_{m \in M^\gamma}$ be $h^\gamma(u^\gamma, x^\gamma) = \min_{m \in M^\gamma}\left\{\frac{1}{x^\gamma_m} u^\gamma_m\right\} - \delta \max_{m,m' \in M^\gamma}\left|\frac{1}{x^\gamma_m} u^\gamma_m - \frac{1}{x^\gamma_{m'}} u^\gamma_{m'}\right|$, where $\delta > 0$ is a small number. This is a Leontief-type welfare function with a slight perturbation, and is continuous in (u^γ, x^γ) (see Figure 3). Let the payoff profile function from an abstract policy $w^\gamma : intX^\gamma \times \mathbb{P} \to \mathbb{R}^{M^\gamma}_{+}$ be such that $w^\gamma(x^\gamma, p) \equiv \arg\max_{u^\gamma \in \bar{V}^\gamma(p)} h^\gamma(u^\gamma, x^\gamma)$. Since $\bar{V}^\gamma$ is a compact-valued and continuous correspondence by Lemma 1, Weierstrass's theorem and Berge's maximum theorem apply, and w^γ is nonempty-valued and upper hemicontinuous. Since $\bar{V}^\gamma(p)$ is comprehensive, $w^\gamma(x^\gamma, p)$ is singleton for any $x^\gamma \in intX^\gamma$, w^γ is a continuous function. The perturbation term of h^γ with $\delta > 0$ ensures the unique solution for this maximization problem. Now, fix $p \in \mathbb{P}$. For all $x^\gamma \in intX^\gamma$, there is only one $u^\gamma \in \partial\bar{V}^\gamma(p) \cap \mathbb{R}^{M^\gamma}_{++}$ is assigned by $w^\gamma(\cdot, p)$. Conversely, for all $u^\gamma \in \partial\bar{V}^\gamma(p) \cap \mathbb{R}^{M^\gamma}_{++}$, $x^\gamma \equiv \frac{u^\gamma}{|u^\gamma|} \in intX^\gamma$ achieves $u^\gamma = \arg\max_{\tilde{u}^\gamma \in \partial\bar{V}^\gamma(p)} h^\gamma(\tilde{u}^\gamma, x^\gamma)$. Thus, we conclude that $w^\gamma(\cdot, p) : intX^\gamma \to \partial\bar{V}^\gamma(p) \cap \mathbb{R}^{M^\gamma}_{++}$ is one-to-one and onto function.■

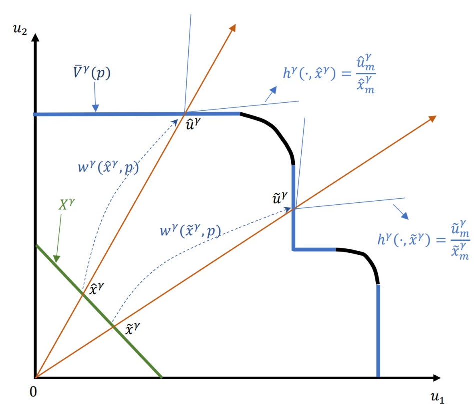

Figure 3

Lemma 3 says that $w^\gamma_m(x^\gamma, p)$ can be regarded as a payoff function of player m from abstract policy x^γ. We will use the w^γ function to assign a weak Pareto payoff vector for each member of team $\gamma \in \Gamma_{team}$. Let $\hat{X}^\gamma \equiv \{x^\gamma \in X^\gamma : x^\gamma_m \geq \epsilon\}$ where $\epsilon > 0$ is small enough that for all γ and p, such that there exists $m = 1, ..., M^\gamma$ such that $w^\gamma_m(x^\gamma, p) < 1 = u^{\gamma^{t(m,\gamma)}}(p)$ holds for all $x^\gamma \in X^\gamma \backslash \hat{X}^\gamma$. Note that $x^\gamma \in X^\gamma \backslash \hat{X}^\gamma$ is an irrelevant abstract policy since at least one player gets less than her individually rational payoff.

For completeness, we describe allocations for (i) $\gamma \notin \Gamma_{team}$, and (ii) $\gamma \in \Gamma_{team}$ but $V^{\gamma}(p) = \{0\}$. Case (i) corresponds to singleton teams $\gamma^t \in \Gamma_{sngl}$. For all $t \in T$, and all $p \in \mathbb{P}$, $u^{\gamma^t}(p) = 1$, and $\bar{\theta}^{\gamma^t}_{WP}(1,p) \equiv \arg\max_{\theta^{\gamma^t} \in Z^{\gamma^t}(p)} u_t(\theta^{\gamma^t}, p)$, which is upper hemi continuous in p. (This mapping assigns policies for each $p \in \mathbb{P}$.) Although the abstract policy for a singleton team is singleton, we let $\hat{X}^{\gamma^t}$ be without truncation: $\hat{X}^{\gamma^t} \equiv \{1\} \equiv X^{\gamma^t} = \Delta^1$ and $w^{\gamma^t}(1,p) = u^{\gamma^t}(p) = 1$, which is a constant mapping (so continuous). Case (ii) is a trivial case that no player wants to choose such team γ.

Theorem 1. Suppose that for all $\gamma \in \Gamma$, and all positions m of team type γ, type $t = t(m,\gamma)$'s indirect payoff function $u^{\gamma}_m(\theta^{\gamma}, p)$ is continuous in (θ^{γ}, p), externality function $\varphi(p,\mu)$ is nonempty-valued and continuous in (p,μ), and feasible policy correspondence $Z^{\gamma}(p)$ is nonempty-valued and continuous in p. Then, there exists an equal-treatment free entry equilibrium under comprehensiveness.

Proof. We prove the theorem by a fixed point theorem. Our fixed point mapping has five components: The first one is a population mapping β^t which assigns type t players to the highest payoff positions for type t, and its Cartesian product $\beta \equiv \Pi_{t\in T}\beta^t$. The second is a policy mapping ϕ^{γ} which assigns the smallest abstract policy to positions that have the highest population in team type γ, and we let its Cartesian product be $\phi \equiv \Pi_{\gamma\in\Gamma}\phi^{\gamma}$. The third is a team-type measure mapping generated from population distribution over the positions of each team type. The fourth is a abstract policy mapping, which assigns individually irrational payoffs to overpopulated positions. The last mapping is a simple price determination mapping.

We start with a population mapping for type $t \in T$. Consider any given profile of abstract policies adopted by teams $x = (x^{\gamma})_{\gamma\in\Gamma}$ together with any price p. Note that we list all team types $\gamma \in \Gamma$ even if their measure is zero. Consider a mapping from a pair of abstract policy profile x and externality variable p to a payoff profile for all team types and positions $u = (u^{\gamma}_m)_{(m,\gamma)\in\mathcal{M}}$. Lemma 3 assures that $w : \prod_{\gamma\in\Gamma} \hat{X}^{\gamma} \times \mathbb{P} \to \mathbb{R}^{\mathcal{M}}_+$ is a continuous function. We partition w based on player type assigned to each position of a team type: a list of payoffs from positions for each type $t \in T$ is described by

$$w^t(x,p) = (w^{\gamma}_m(x^{\gamma}, p))_{(m,\gamma)\in\mathcal{M}^t}$$

where $\mathcal{M}^t = \{(m,\gamma) : \alpha^{\gamma}(m) = t \text{ for some } \gamma \in \Gamma, (m,\gamma) \in \mathcal{M}^{\gamma}\}$. Note that $\cup_{t\in T}\mathcal{M}^t = \cup_{\gamma\in\Gamma}\mathcal{M}^{\gamma}$ holds.

Let $\beta^t : \prod_{\gamma\in\Gamma} \hat{X}^{\gamma} \times \mathbb{P} \to \left(\bar{\nu}^t \Delta^{\mathcal{M}^t}\right)$ be population mapping of type t players such that

$$\beta^t(x,p) = \{n^t \in \bar{\nu}^t \Delta^{\mathcal{M}^t} : n^t \in \arg\max_{n^t} \sum_{(m,\gamma)\in\mathcal{M}^t} n^t_{(m,\gamma)} w^{\gamma}_m(x^{\gamma}, p)\}$$

where $n^t_{(m,\gamma)} \equiv \nu_{(m,\gamma)}(\Theta^{\gamma})$ is the measure of type t players assigned to position $(m,\gamma) \in \mathcal{M}^t$, and $\sum_{(m,\gamma)\in\mathcal{M}^t} n^t_{(m,\gamma)} = \bar{\nu}^t$. Since $\sum_{(m,\gamma)\in\mathcal{M}^t} n^t_{(m,\gamma)} w^{\gamma}_m(x^{\gamma}, p)$ is a continuous function in (x^{γ}, p, n^t), mapping β^t is nonempty-valued, upper hemicontinuous, and convex-valued. By construction, $\beta^t(x,p)$ assigns type t players' population to the positions bringing them the highest equilibrium payoff. Taking the Cartesian product across types $\beta(x,p) = \prod_{t\in T}\beta^t(x,p)$, we have our population mapping $\beta : \prod_{\gamma\in\Gamma} X^\gamma \times \mathbb{P} \to \prod_{t\in T}\left(\bar{\nu}^t\Delta^{\mathcal{M}^t}\right)$, which is nonempty-valued, upper hemicontinuous, and convex-valued.

Now, we turn to our abstract policy mapping for each $\gamma \in \Gamma$. Let $n^\gamma \in \mathbb{R}_+^{\mathcal{M}^\gamma}$ be a population distribution in γ. In a feasible assignment, we need to have $n_m^\gamma = n_{m'}^\gamma$ for all $m, m' \in \mathcal{M}^\gamma$. Towards this goal, for each $\gamma \in \Gamma$, define $\phi^\gamma : \mathbb{R}_+^{\mathcal{M}^\gamma} \to \hat{X}^\gamma$ such that

$$\phi^\gamma(\nu^\gamma) = \left\{x^\gamma \in \hat{X}^\gamma : x^\gamma \in \arg\min_{x^\gamma} \sum_{m\in\mathcal{M}^\gamma} x_m^\gamma n_m^\gamma\right\},$$

where ϕ^γ mapping assigns $x_m^\gamma = \epsilon$ for the most populated positions m unless $n_m^\gamma = n_{m'}^\gamma$ holds for all $m, m' \in \mathcal{M}^\gamma$ (otherwise, $\phi^\gamma(\nu^\gamma) = \hat{X}^\gamma$ holds).[22] Since $\sum_{m=1}^{M^\gamma} x_m^\gamma n_m^\gamma$ is a continuous function in (x^γ, n^γ), mapping ϕ^γ is nonempty-valued, upper hemicontinuous, and convex-valued. Taking the Cartesian product across types $\phi(\nu) = \prod_{\gamma\in\Gamma}\phi^\gamma(n^\gamma)$, we have our abstract policy mapping $\phi : \prod_{t\in T}\left(\bar{\nu}^t\Delta^{\mathcal{M}^t}\right) \twoheadrightarrow \prod_{\gamma\in\Gamma} X_\epsilon^\gamma$, which is nonempty-valued, upper hemicontinuous, and convex-valued.

Next, we will construct a team measure mapping over policies χ^γ. To do that, first let $\Delta^\Gamma \equiv \left\{(t^\gamma)_{\gamma\in\Gamma} \in \mathbb{R}_+^\Gamma : \sum_{\gamma\in\Gamma} t^\gamma = 1\right\}$ and let $\tau : \prod_{t\in T}\left(\bar{\nu}^t\Delta^{\mathcal{M}^t}\right) \twoheadrightarrow \left(\max_{t\in T}\bar{\nu}^t\right)\Delta^\Gamma$ be such that $\tau(n) = \left(\tau^\gamma\left(n^\gamma\right)\right)_{\gamma\in\Gamma} = \left(\max_{m\in\mathcal{M}^\gamma} n_m^\gamma\right)_{\gamma\in\Gamma}$, which describes the measure of each team γ that can accommodate its population distribution $n^\gamma = (n_m^\gamma)_{m\in\mathcal{M}^\gamma}$. Clearly, it is a continuous function.

Second, we need to connect an abstract policy x of team γ with actual policies θ^γ (see Figure 4). Note that w^γ maps x^γ to payoff space one-to-one. Let $\tilde{\theta}^\gamma : X_\epsilon^\gamma \times \mathbb{P} \twoheadrightarrow \Theta^\gamma$ be $\tilde{\theta}^\gamma(x^\gamma, p) \equiv \bar{\theta}_{WP}(w^\gamma(x^\gamma,p),p)$ for all $\gamma \in \Gamma_{team}$, and $\tilde{\theta}^{\gamma^t}(x^{\gamma^t}, p) \equiv \bar{\theta}_{WP}^{\gamma^t}(1,p)$ for $\gamma^t \in \Gamma_{sngl}$. Mapping $\tilde{\theta}^\gamma$ maps abstract policy x^γ to a subset of policies that supports payoff vector $w^\gamma(x^\gamma,p)$. Since mapping $\bar{\theta}_{WP}$ is upper hemicontinuous (Lemma 2), and w^γ is a continuous function, $\tilde{\theta}^\gamma$ is upper hemicontinuous.

Now, we can construct team measure mapping over policies. Let $\mathbb{M}^\gamma$ be the set of measurable functions on Θ^γ bounded above by $\max_{t\in T}\bar{\nu}^t$, and let $\chi^\gamma : \left(\max_{t\in T}\bar{\nu}^t\right)\Delta^\Gamma \times X_\epsilon^\gamma \times \mathbb{P} \twoheadrightarrow \mathbb{M}^\gamma$ be $\chi^\gamma(n^\gamma, x^\gamma, p) = \left\{\mu^\gamma \in \mathbb{M}^\gamma : \mu^\gamma\left(\Theta^\gamma\right) = \mu^\gamma\left(\tilde{\theta}^\gamma(x^\gamma,p)\right) = \tau^\gamma(n^\gamma)\right\}$. Let $\chi : \left(\max_{t\in T}\bar{\nu}^t\right)\Delta^\Gamma \times \prod_{\gamma\in\Gamma} X_\epsilon^\gamma \times \mathbb{P} \twoheadrightarrow \prod_{\gamma\in\Gamma}\mathbb{M}^\gamma$ be a Cartesian product of χ^γs. Since $\tilde{\theta}^\gamma$ is nonempty-valued, and upper hemicontinuous, χ^γ and χ are nonempty-valued, convex-valued, and upper hemicontinuous, too (see Mas-Colell 1984).

[22] Note that $x_m^\gamma = \epsilon$ means that $u_m^\gamma(\theta^\gamma, p) < u^{\gamma^{t(m,\gamma)}}(p) = 1$ for $\theta^\gamma \in \tilde{\theta}^\gamma(x^\gamma, p)$. That is, if the unbalanced population in team type γ cannot be a part of a fixed point, since over populated position's type would receive a payoff less than individually rational one. This is related to a price mapping in the Gale-Nikaido mapping (see Debreu, 1959; for a recent comprehensive treatment of this approach, see Khan, McLean, and Uyanik, 2025).

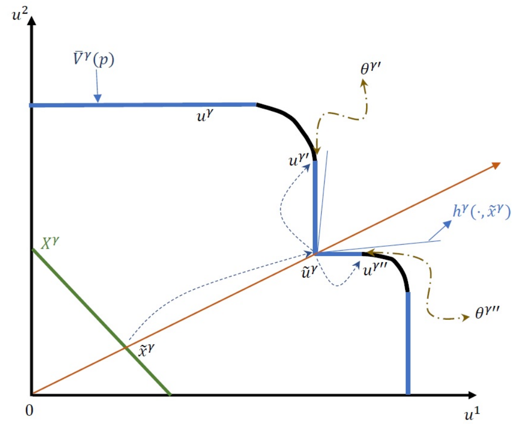

Figure 4

Lastly, the externality mapping $\varphi : \prod_{\gamma\in\Gamma} \mathbb{M}^{\gamma} \times \mathbb{P} \to \mathbb{P}$ is the final block of our fixed point mapping $p = \varphi(\tau, p)$. It determines the externality variables level under (ν, p). This is a continuous function by assumption.

Our fixed point mapping $\Phi : \prod_{t\in T} \left(\bar{\nu}^t \Delta^{\mathcal{M}^t}\right) \times \prod_{\gamma\in\Gamma} X^{\gamma}_{\epsilon} \times \prod_{\gamma\in\Gamma} \mathbb{M}^{\gamma} \times \mathbb{P} \twoheadrightarrow \prod_{t\in T} \left(\bar{\nu}^t \Delta^{\mathcal{M}^t}\right) \times \prod_{\gamma\in\Gamma} X^{\gamma}_{\epsilon} \times \prod_{\gamma\in\Gamma} \mathbb{M}^{\gamma} \times \mathbb{P}$ is a Cartesian product of mappings β, ϕ, χ, and φ. This mapping has a nonempty, compact, and convex set as its domain and range, and the mapping is nonempty-valued, upper hemicontinuous, and convex-valued. Thus, by Fan-Glicksburg's fixed point theorem, there exists a fixed point $(n, x, \mu, p) \in \prod_{t\in T} \left(\bar{\nu}^t \Delta^{\mathcal{M}^t}\right) \times \prod_{\gamma\in\Gamma} X^{\gamma}_{\epsilon} \times \prod_{\gamma\in\Gamma} \mathbb{M}^{\gamma} \times \mathbb{P}$.

The rest of the proof shows that the fixed point (n, x, μ, p) corresponds to an equal-treatment free entry equilibrium under comprehensiveness. First notice that for all $\gamma \in \Gamma$, and all $m, m' = 1, ..., M^{\gamma}$, $n^{\gamma}_m = n^{\gamma}_{m'}$ must hold. It is shown by contradiction. Suppose that for some γ, $n^{\gamma}_m > n^{\gamma}_{m'}$ holds. Without loss of generality, let position m be the highest population among all positions in team type γ. Then, by construction of ϕ^{γ}, $x^{\gamma}_m = \epsilon$ holds. This implies that $u^{t(m,\gamma)} < 1 = u^{\gamma^t}$ despite that her outside option achieves a positive payoff. By the construction of $\beta^{t(m,\gamma)}$, $n^{\gamma}_m = 0$ must follow. This is a contradiction. Thus, for all $\gamma \in \Gamma$, $n^{\gamma}_1 = ... = n^{\gamma}_{M^{\gamma}}$ holds. Recalling that the distribution of type $t(m,\gamma)$ players over team type γ's policies Θ^{γ} is described by a measure ν^{γ}_m over Θ^{γ}, we let all positions $m \in \mathcal{M}^{\gamma}$ have the same distribution over Θ^{γ}: $\nu^{\gamma}_1 = ... = \nu^{\gamma}_{M^{\gamma}} = \mu^{\gamma}$ for all $\gamma \in \Gamma$. This makes (ν, μ) a feasible assignment. By the construction of β, all types of players are choosing the highest payoff positions at the fixed point, so for all $t \in T$, $\beta^t_{(m,\gamma)}(x, p) > 0$ implies $w^{\gamma}_m(x^{\gamma}, p) \geq w^{\gamma'}_{m'}(x^{\gamma'}, p)$ for all $(\gamma', m') \in \mathcal{M}^{t(\gamma,m)}$ for all $m = 1, ..., M^{\gamma}$. For all $t \in T$, let $u^*_t \equiv w^{\gamma}_m(x^{\gamma}, p)$ for all $(m,\gamma) \in \mathcal{M}^t$ with $\beta^t_{(m,\gamma)}(x, p) > 0$. Since x^{γ} is a weakly Pareto efficient for all γ, there is no strictly improving coalitional deviation. By the construction of χ^{γ} mapping, we have $\mu^{\gamma}(\Theta^{\gamma}) = \mu^{\gamma}\left(\tilde{\theta}^{\gamma}(x^{\gamma}, p)\right)$ for all $\gamma \in \Gamma$. This implies that for all $\gamma \in \Gamma$ with

$\mu^\gamma(\Theta^\gamma) > 0$, and all $\theta^\gamma \in \tilde{\theta}^\gamma(x^\gamma, p) \subset \Theta^\gamma_{WP}(p)$, $u^\gamma_m(\theta^\gamma, p) \geq w^\gamma_m(x^\gamma, p) = u^*_{t(m,\gamma)}$ holds for all $m = 1, ..., M^\gamma$ (see Figure 4). Thus, letting $\nu^\gamma_m(\theta^\gamma) = \mu^\gamma(\theta^\gamma)$ for all $m \in \mathcal{M}^\gamma$ and all $\gamma \in \Gamma$, $\left(p, (u^*_t)_{t\in T}, (\nu^\gamma_m)_{m\in\mathcal{M}^\gamma,\gamma\in\Gamma}, (\mu^\gamma)_{\gamma\in\Gamma}\right)$ is an equal-treatment free entry equilibrium under comprehensiveness.□

This result immediately implies the following theorem.

Theorem 2. Suppose that for all $\gamma \in \Gamma$, and all positions m of team type γ, type $t = t(m, \gamma)$'s indirect payoff function $u^\gamma_m(\theta^\gamma, p)$ is continuous in (θ^γ, p), externality function $\varphi(p, \mu)$ is continuous in (p, μ), and feasible policy correspondence $Z^\gamma(p)$ is nonempty-valued and continuous in p. Then, there exists a free entry equilibrium.

Proof. From the last part of the proof of Theorem 1, we know that for all $\gamma \in \Gamma$ with $\mu^\gamma(\Theta^\gamma) > 0$, and all $\theta^\gamma \in \tilde{\theta}^\gamma(x^\gamma, p) \subset \Theta^{WP}(p)$, $u^\gamma_m(\theta^\gamma, p) \geq w^\gamma_m(x^\gamma, p) = u^*_{t(m,\gamma)}$ holds for all $m = 1, ..., M^\gamma$. Thus, only zero measure of players are not assigned to weakly Pareto efficient policies, and almost all players are distributed over the weak Pareto policies according to μ^γ, achieving $(u^*_t)_{t\in T}$ or higher payoffs in the associated feasible allocation. However, the above allocation is an equal-treatment free entry equilibrium under comprehensiveness, and there is no strictly improving diation from it. This implies that the associated feasible allocation is immune to strictly improving deviation as well. We completed the proof.□

5 Concluding Remarks: Some Applications

The advantage of this approach is that we do not need to select a (Pareto) optimal allocation for a club or a household. The second best set of allocations may be highly nonconvex, but in payoff space, it could be much more well-behaved. By inventing the way to keep track of feasible allocations that achieve a payoff vector as an inverse mapping of payoff functions, we only need continuity of payoff functions and the compactness of an implementable set in payoff space. To conclude, we provided several applications of our model.

5.1 Labor Managed Firms: Revisited

Here, we show how our general framework can accommodate the leading example presented in Section 2. The market price p is determined by an inverse demand function $p = P(Y)$, where Y is aggregate output of the product. We assume that $\mathbb{P} \equiv [0, \bar{p}]$, and $P : \mathbb{R}_+ \to \mathbb{P}$ is a continuous and nonincreasing function. Since each team is atomless, market price $P(Y)$ is taken as given by each player and team. teams can engage in production by hiring players who can provide heterogeneous labor depending on their types (distinguished by their ability, marginal cost of effort etc). Within each team players engage in effort contribution game in producing the product, and the revenue of producing the product is fully distributed among the team players (labor managed firm). For example, we can consider a simple proportional revenue sharing rule as Θ^γ.[23] Each team type $\gamma \in \Gamma$ chooses a proportional revenue sharing rule $\theta^\gamma = (\theta_1^\gamma, ..., \theta_{M^\gamma}^\gamma) > 0$ with $\sum_{m=1}^{M^\gamma} \theta_m^\gamma = 1$, thus $\theta^\gamma \in \Theta^\gamma \equiv \Delta^{M^\gamma}$, where Δ^S is an S-dimensional simplex. Consider type $t = t(m,\gamma)$ player hired in position m of team γ. Type $t \in T$ player has a quasi-linear payoff function, and solves the following optimization problem

$$\max_{e_m^\gamma} u_t = \theta_m^\gamma p f^\gamma(e_1^\gamma, ..., e_m^\gamma, ..., e_{M^\gamma}^\gamma) - C^t(e_m^\gamma),$$

where $f^\gamma(\cdot)$ is team γ's production function, and $C^t(e)$ is type t player's effort cost function. Both θ_m^γ and p are taken as exogenous in the player's optimization problem. Equilibrium payoff of position m player (type $t = t(m,\gamma)$) and equilibrium output are described by $u_m^\gamma(\theta^\gamma, p)$ and $y^\gamma(\theta^\gamma, p)$, and the distribution of production teams over policies in team type γ Θ^γ is μ^γ. Then, $Y(p,\mu) = \sum_{\gamma\in\Gamma} \int_{\Theta^\gamma} y^\gamma(\theta^\gamma, p) d\mu^\gamma$, and by setting $\varphi(p,\mu) = P(Y(p,\mu))$, we can embed this example in our abstract problem.

5.2 Endogenous Households and Markets

Gersbach, Haller, and Konishi (2015) analyzed a general equilibrium model with large number of consumers/agents when households (composed of a husband and a wife) are endogenously formed and the family members choose their consumption vectors and actions jointly, which are subject to local externalities within the family. Their concerns are the existence of equilibrium and its efficiency. Our approach of working on payoff space instead of policy/allocation space can generalize their results significantly. Here, we illustrate how we can embed their model into our model. Following the literature, we assume that there are two group of consumers, male and female, and a married household can be formed by a pair of male and female consumers.[24] Let $m \in T^M$ and $w \in T^W$ be a representative couple (types) of male and female consumers. There are L private goods and some discrete action set A that are subject to externalities within the household, and each couple (m,w) chooses their policy $(x_m, x_w, a) = \theta \in \Theta \equiv \mathbb{R}_+^L \times \mathbb{R}_+^L \times A$ collectively. A matching is described as a measure μ over $T^M \times T^W \times \Theta$ with measure consistency. Assuming that there is a commodity with strong monotonicity for all consumers, any household exhaust their budget, and as a result, the Walras Law is satisfied. Letting $\mathbb{P} \equiv \Delta^L \times Z$, where Δ^L is a price simplex and $Z \subset \mathbb{R}^L$ is a set of excess demand, which is compatified with a standard method. The only difference from our theorem is that we need the externality mapping to be a correspondence instead of a continuous function: Deriving the aggregated excess demand by integrating the consumption bundles with measure μ, we can construct a price-determining fixed point mapping φ, and prove the existence of stable (divorce-free) household distribution in a market equilibrium and its optimality as in Haller et al. (2015) through the Gale-Nikaido's

[23] Ichiishi (1977) considered labor managed firms withendogeneous memberships in a strictly finite model. He identified a set of sufficient conditions that guarantee a free entry equilibrium without asymmetric information.

[24] We can drop this assumption at no cost as is seen from our analysis.

lemma (see Debreu 1959).[25] However, our method has a strong advantage over their method. They work on a fixed point mapping of a policy (collective choice of consumption vectors and (unpriced) actions), which requires the convexity of their preferences—they assume quasiconcavity of payoff functions for a pair of consumption vectors to find a Pareto efficient allocation by using the method in Shafer and Sonnenschein (1975). In contrast, in this paper, we work on payoff space instead of allocation space, and we can drop quasiconcavity of payoff function completely, and we can also allow for suboptimal allocations for a married couple (due to lack of pre-marriage commitment power etc.). We can also apply the same method for club and local public good economies as in Ellickson et al. (1998), Allouch et al. (2009), and Konishi (2010, 2013).[26]

5.3 Large Group Contests with Random Matching

Konishi, Pan, and Simeonov (2025) consider a Tullock contest played in an K team sport league, in which each team has M positions and a fixed sharing rule is set by a social norm. It is still hard to show a general existence of stable team structure due to integer problems. Here, we illustrate how our approach provides a remedy by replicating the leagues. In their model, players differ in their type $t \in T$ with heterogeneous ability $a_t > 0$, and make effort contributions e with a common linear cost function $c(e) = e$. A representative team type $\gamma \in \Gamma$ is an assignment function $\alpha^\gamma : \{1, ..., M\} \to T\}$ with a common technology f. Konishi, Pan, and Simeonov (2025) consider $K \times M$ players form K teams, and compete for a single prize with a Tullock team contest: i.e., team k's winning probability is $\pi_k(y_1, ..., y_K) = \frac{y_k}{\Sigma_{k'=1}^K y_{k'}}$, through their team output $y'_k \geq 0$ produced by an identical CES production function $f(e_1, ..., e_M) = \left(\Sigma_{m=1}^M (a_m e_m)^\sigma\right)^{\frac{1}{\sigma}}$, where $a_m = a_{t(\gamma,m)}$ and $e_m = e_{t(\gamma,m)}$ are the ability and effort of a player who occupies position m. The expected payoff of a player of team k (of some type γ with a prize-sharing rule $\theta^k = (\theta_1^k, ..., \theta_M^k) \in \Delta^M = \Theta$) in position m is $\tilde{u}_m^k = \theta_m^k \pi^k(y_1, ..., y_M) - e_m$. If a league is characterized by the profile of team types and their policies $\left(\gamma^k, \theta^k\right)_{k=1,...,K}$, its equilibrium payoff profile for all teams and all positions is written as $\left(u_m^k\left(\left(\gamma^h, \theta^h\right)_{h=1,...,K}\right)\right)_{k=1,...,K;m=1,...,M}$.

Now we replicate this problem by offering a prize for each league and consider a continuum of leagues of K teams. The distribution of teams is described by measure μ on $\Gamma \times \Theta$. Assuming the law of large numbers in continuum of random variables (Judd 1985), we consider a team with (γ, θ) play a league with $K - 1$ teams randomly drawn from μ. With this machinery, team type distribution

[25] We cannot directly apply our proof to this problem, since the price-determination mapping φ in the Gale-Nikaido lemma needs to be a correspondence, while our externality mapping φ is assumed to be single-valued. However, it is easy to show that market excess demand and Walrasian auctioneer mappings are nonempty-valued, upper hemicontinuous, and convex-valued under the standard conditions, so we can apply the same fixed point mapping by calling a Cartesian product of the market excess demand and the Walrasian auctioneer mappings φ.

[26] Recently, Konishi (2025) applied the method proposed here to urban economics model to generalize Konishi (2013).

μ determines the likelihood of each league type realizations, determining each type player's expected payoff by belonging her team. Then, the expected payoff of type $t(m, \gamma)$ who occupies position m of team-type γ with policy θ when team distribution is described by measure μ can be written as

$$U_m\left((\gamma, \theta); \mu\right) \equiv \int_\Theta \left[\int_\Theta ... \left[\int_\Theta u_m\left((\gamma, \theta); \left(\gamma^h, \theta^h\right)_{h=2,...,K}\right) d\mu\right] ... d\mu\right] d\mu.$$

Using this, we can define comprehensive cover of feasible payoff vectors of team type γ under μ by $\bar{V}^\gamma(\mu)$. This way, we can embed this random matching problem into our framework, and we can prove the existence of free entry equilibrium. Existing distribution of team types is certainly important in determining which team choose which policies.[27] Chade and Eeckhout (2020) considers a random matching model without action choice, which can be considered as a special case of the above mentioned application when policy choices and effort contribution games are abstracted.

Appendix

Here we assume a CES technology $f(e_1, e_2) = (a_1 e_1^\sigma + a_2 e_2^\sigma)^{\frac{\rho}{\sigma}}$ with $\sigma \in (0,1)$, and calculate Nash equilibrium of an effort contribution game with heterogeneous shares in our labor managed firm example. Workers' cost of making effort is linear: $c_i(e_i) = e_i$ for $i = 1, 2$. Without loss of generality, we assume $\bar{\nu}_1 = 1$ and $\bar{\nu}_2 = 2$. Each team can choose its proportional sharing rule (θ_1, θ_2).

In a team with (θ_1, θ_2), worker $i = 1, 2$ solves the following problem:

$$\max_{e_i} \theta_i p f(e_i, e_j) - e_i.$$

Let $y = f(e_1, e_2) = (a_1 e_1^\sigma + a_2 e_2^\sigma)^{\frac{\rho}{\sigma}}$. The first order conditions are for $i = 1, 2$

$$\frac{\partial f}{\partial e_i} = \theta_i p \frac{\rho a_i}{e_i^{1-\sigma}} y^{\frac{\rho-1}{\rho}} - 1 = 0,$$

or

$$e_i = (\theta_i a_i p \rho)^{\frac{1}{1-\sigma}} y^{\frac{\rho-\sigma}{\rho(1-\sigma)}}.$$

Substituting them back to $y = (a_1 e_1^\sigma + a_2 e_2^\sigma)^{\frac{\rho}{\sigma}}$, we obtain

$$y = \left(a_1^{\frac{1}{1-\sigma}} \theta_1^{\frac{\sigma}{1-\sigma}} + a_2^{\frac{1}{1-\sigma}} \theta_2^{\frac{\sigma}{1-\sigma}}\right)^{\frac{\rho(1-\sigma)}{\sigma(1-\rho)}} p^{\frac{\rho}{1-\rho}} \rho^{\frac{\rho}{1-\rho}}.$$

Thus, we can write e_i as a function of economic data.

$$e_i = \theta_i^{\frac{1}{1-\sigma}} a_i^{\frac{1}{1-\sigma}} \left(a_1^{\frac{1}{1-\sigma}} \theta_1^{\frac{\sigma}{1-\sigma}} + a_2^{\frac{1}{1-\sigma}} \theta_2^{\frac{\sigma}{1-\sigma}}\right)^{\frac{\rho-\sigma}{\sigma(1-\rho)}} p^{\frac{1}{1-\rho}} \rho^{\frac{1}{1-\rho}}$$

[27] In Konishi et al., they consider an exogenous fixed sharing rule, corresponding to the case of $|\Theta = 1|$ and μ being a vector of scalars. However, our approach extends to the case with a general compact Θ in which each team can choose θ.

Substituting them back to $u_i = \theta_i p y - e_i$, worker i's intra-team equilibrium payoff is written as

$$u_i = \left\{ \theta_i \left(a_1^{\frac{1}{1-\sigma}} \theta_1^{\frac{\sigma}{1-\sigma}} + a_2^{\frac{1}{1-\sigma}} \theta_2^{\frac{\sigma}{1-\sigma}} \right)^{\frac{\rho(1-\sigma)}{\sigma(1-\rho)}} \rho^{\frac{\rho}{1-\rho}} - \theta_i^{\frac{1}{1-\sigma}} a_i^{\frac{1}{1-\sigma}} \left(a_1^{\frac{1}{1-\sigma}} \theta_1^{\frac{\sigma}{1-\sigma}} + a_2^{\frac{1}{1-\sigma}} \theta_2^{\frac{\sigma}{1-\sigma}} \right)^{\frac{\rho-\sigma}{\sigma(1-\rho)}} \rho^{\frac{1}{1-\rho}} \right\} p^{\frac{1}{1-\rho}}.$$

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